Tauberian Theorems for Generalized Functions

Mathematics and Its Applications (*Soviet Series*)

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EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders.*

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet Union, Eastern Europe, and Japan, it was decided to devote special attention to the work emanating from these particular regions. Thus it was decided to start three regional series under the umbrella of the main MIA programme.

Tauber theorems, the topic of this volume in the MIA (USSR) series, are rather difficult to place in the general scheme of things mathematical I have always found. They are theorems which connect the asymptotic behaviour of a (generalized) function in the neighborhood of zero with the behaviour of its Fourier- or Laplace- (or other integral-) transform near infinity. That should put them somewhere in the section on integral transforms, possibly Fourier transforms. But then the applications: these range from statistics to number theory from harmonic analysis, filtering and signal processing to differential equations and mathematical physics. All this is for one variable Tauberian theory.

For more variable Tauberian theory the applications have, until fairly recently, not amounted to very much. Quite possibly because the available theory did not go beyond rather trivial straightforward generalizations. On the other hand, many of the applications areas above have natural more dimensional extensions; think of filtering and prediction, for example, or statistics, or ... Other demands came from mathematical physics (both for Tauberian theory for several variables and for distributions in one and more variables) and, as the authors report in their preface, the subject took off: both in theory and applications. This is the first and only book on the subject which is no wonder given the origin of the main theorems. It seems more than likely to spawn many more applications, e.g. to the more variable or distributional cases of the areas listed for the one-variable case.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, October 1986

Michiel Hazewinkel

PREFACE

Tauberian theorems are usually assumed to connect the asymptotic behaviour of a generalized function (or a distribution) in the neighbourhood of zero with that of its Fourier transform, Laplace transform, or some other integral transform at infinity. The inverse theorems are usually called 'Abelian'.

The term 'Tauberian theorems' was introduced to mathematics by A. Tauber in 1897. He proved the first theorem of this type and all such theorems are now known by the general name 'Tauberian theorems'. Tauberian theory was intensively developed in the first half of this century and the results were gathered by N. Wiener in his extensive work 'Tauberian theorems' (1932) and in **Divergent series** by G.H. Hardy (1949).

For the case of only one variable, and for measures, Tauberian theory is rather advanced. It has many applications in number theory, in harmonic analysis, in probability theory, in differential equations, and in mathematical physics. One can find a survey of the main results and literature in A.G. Postnikov's 'Tauberian Theory and its Applications' (1979), in M.A. Subhankulov's **Tauberian Theorems with Remainder** (1979), and in T.H. Ganelius' 'Tauberian remainder theorems' (1971).

To date, in contrast with the one-dimensional case, many-dimensional Tauberian theory was no essential application and its development is characterized by straightforward and more or less trivial generalizations of one-dimensional results to a many-dimensional situation. In this direction only a few results related to multiple series or measures, with support in a positive octant, have been obtained.

The requirements of modern mathematics, and especially of mathematical physics, make it imperative that a Tauberian theory for the distributions (generalized functions) of many variables should be developed. In other words, it is necessary to extend the classical Tauberian theory to the case of more general objects - namely, for distributions defined on a many-dimensional space. This problem was initiated in the work of N.N. Bogolyubov, V.S. Vladimirov and A.N. Tavkhelidze (1972), in connection with a theoretical explanation of experimentally observed phenomena - the so-called automodel behaviour in quantum field theory, typical, say, for deep inelastic lepton-hadron scattering (Fig. 1).

This paper was followed by a systematic study of the many-dimensional Tauberian theory for distributions, and of its applications, in the Department of Mathematical Physics at the Steklov Mathematical Institute. In 1973, B.I. Zavialov introduced the important concept of the quasi-asymptotics of distributions and applied it to a study of the asymptotic properties of form factors and their Jost-Lehmann-Dyson spectral functions. In 1976, V.S. Vladimirov gave a many-dimensional generalization of the well-known Hardy-Littlewood Tauberian theorem for measures located on a nonnegative half axis. In a series of papers (1977-1985) by Yu.N. Drozzinov and B.I. Zavialov, the many-dimensional theorem

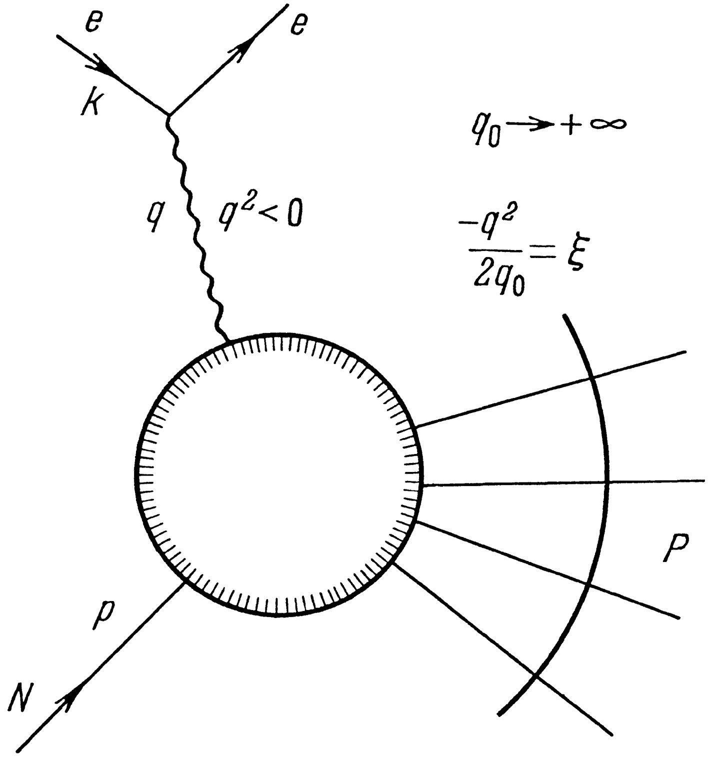

Figure 1.

of Vladimirov was extended to some classes of distribution, and the notion of quasi-asymptotics was shown to be the most suitable tool for the formulation of Abelian and Tauberian distribution theorems.

These results will be discussed in Chapter 2 (the many-dimensional case) and in Chapter 3 (the one-dimensional case). In Chapter 2 we also formulate the so-called 'comparison' Tauberian theorems (for example, by Karamata and Keldysh) on the basis of a many-dimensional generalization of regularly varying functions. In Chapter 3 some of the many-dimensional results of Chapter 2 are reformulated for the one-dimensional case in a more precise and detailed fashion. We also prove a theorem on the decomposition of a tempered distribution, defined on the whole axis, into a sum of distributions from $\mathcal{S}'_+$ and $\mathcal{S}'_-$, which preserves the quasi-asymptotic properties. Some applications of this theorem are also included.

In a series of papers (1979-1981), V.S. Vladimirov and B.I. Zavialov extended the results on the automodel behaviour of form factors to arbitrary causal (distributions) functions and expressed these results in the form of Tauberian and Abelian theorems. Vladimirov and Zavialov connected the automodel behaviour of a causal function with the asymptotic behaviour of its Fourier transform in the neighbourhood of a light cone. It is shown that in a description of the automodel behaviour of a causal function the fundamental role is played by the Radon transform of distributions, with support in a three-dimensional ball. This, in turn, implies the so-called 'sum rules' for limit distributions. All these results are discussed in Chapter 5, where we also give a more precise version and new proof of the well-known Jost-Leymann-Dyson representation, study the questions of uniqueness, and give the inversion formulae of this representation (Section 13).

Among other applications of the many-dimensional Tauberian theory for distributions we mention only the following: the Abel and Česaro summation of divergent series with respect to an automodel weight (Section 9); the asymptotic behaviour of the Cauchy problem solution for parabolic and hyperbolic equations and that of the fundamental solutions of linear passive systems (Chapter 4); the many-dimensional generalizations of the Lindelöf theorem in the complex analysis of several variables (see Yu.N. Drozzinov and B.I. Zavialov (1977)); the study of the Bellman-Harris branch processes (see A.L. Yakymiv (1981, 1983)), and (Section 5); the study of many-dimensional spectral asymptotics for elliptic operators (see S.M. Kozlov (1983)); the asymptotic properties of the two-point Wightman function (Section 14); and in statistical physics, the existence of spontaneous magnetization in the Ising model, which is a typical Tauberian condition, implies some additional restrictions on the Lee-Yan measure (see V.S. Vladimirov, and I.V. Volovich (1982)).

In this book we do not touch the so-called Tauberian theorems with the remainder term, but refer the reader to the works of M.A. Subhankulov (1976), T. Ganelius (1971), L. Frennemo (1965, 1966) and K.A. Bukin (1981).

All the necessary tools for the theory of distributions and related topics in analsysis, that are used in this book are given in Chapter 1, where, in particular, we give the theory of the space $\mathcal{S}'(F)$, where F is a closed regular set.

As well as the more traditional aspects of distribution theory, we discuss in this book in more detail some integral transformations of distributions (such as Fourier transform, Laplace transform, Radon transform, B-transform, fractional derivation, integration with respect to a regular cone, and so on). We also discuss questions related to the quasi-asymptotic properties of distributions with respect to a given automodel function and to give family of automorphisms of a cone.

The Authors

Chapter 0

NOTATION AND DEFINITIONS

0.1 We denote the *points* of an n-dimensional real space $\mathbf{R}^n$ by $x, y, \xi, \eta, p, q, \ldots$; $x = (x_1, x_2, \ldots, x_n)$. The points of an n-dimensional complex space $\mathbf{C}^n$ are given by $z, \zeta, \ldots$; $z = (z_1, z_2, \ldots, z_n) = x + iy$; $x = \operatorname{Re} z$ is the real part of z and $y = \operatorname{Im} z$ is the imaginary part of z, $\bar{z} = x - iy$ is the *complex conjugate* of z; $\zeta = p + iq$. We introduce in $\mathbf{R}^n$ and $\mathbf{C}^n$ the *scalar products*

$$(x, \xi) = x_1\xi_1 + \cdots + x_n\xi_n, \quad \langle z, \zeta\rangle = z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n$$

and the *norms*

$$|x| = \sqrt{(x, x)} = (x_1^2 + \cdots + x_n^2)^{1/2},$$

$$|z| = \sqrt{\langle z, z\rangle} = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}.$$

0.2. *Sets in* $\mathbf{R}^n$ are denoted by $A, B, \mathcal{O}, G, \ldots$; ϕ is the *empty* set. We denote $\bar{A}$ the *closure* of a set A in $\mathbf{R}^n$ (in $\mathcal{O}$), by $\operatorname{int} A$ the set of *interior points* of A, by $\partial A = \bar{A} \setminus \operatorname{int} A$ the *boundary* of A. We shall say that a set A is *compact* in a set B (or is *strictly contained* in B) if A is bounded and $\bar{A} \subset B$; then we write $A \subset\subset B$.

Let A and B be sets in $\mathbf{R}^n$. We shall denote by $A + B$ the set of all $\xi \in \mathbf{R}^n$ such that $\xi = x + y$, $x \in A$, $y \in B$; that is,

$$A + B = [\xi \in \mathbf{R}^n : \xi = x + y, x \in A, y \in B].$$

The following notation is used:

$$B(x_0; R) = [x : |x - x_0| < R]$$

is an *open ball* of radius R with centre at the point x_0; $S(x_0; R) = \partial B(x_0; R)$ is a *sphere* of radius R with centre at the point x_0; $B_R = B(O; R)$, $S_R = S(O, R)$, $S_1 = S^{n-1}$.

We use $\Delta(A, B)$ to denote the *distance* between the sets A and B in $\mathbf{R}^n$; that is,

$$\Delta(A, B) = \inf_{x \in A, y \in B} |x - y|.$$

In the case $B = \partial A$ and $x \in A$ we write $\Delta(x, \partial A) = \Delta_A(x)$.

We use A^ϵ to denote the *ϵ-neighbourhood* of a set A; that is,

$$A^\epsilon = A + B_\epsilon = [x : x = y + \xi, y \in A, |\xi| < \epsilon];$$

A_ϵ denotes the set of those points of A which are separated from ∂A by a distance greater than ϵ; that is,

$$A_\epsilon = [x : x \in A, \Delta_A(x) > \epsilon].$$

The *characteristic function* of a set A is the function $\theta_A(x)$ which is equal to 1 when $x \in A$ and is equal to 0 when $x \bar{\in} A$. The characteristic function $\theta_{[0,\infty)}(x)$ of the semi-axis $x \geqslant 0$ is called the *Heaviside function* and is denoted by $\theta(x)$:

$$\theta(x) = 0, x < 0; \ \theta(x) = 1, x \geqslant 0.$$

We write $\theta_n(x) = \theta(x_1) \cdots \theta(x_n)$; $\epsilon(x) = \operatorname{sign} x = \theta(x) - \theta(-x)$. A set A is called *solid* if $\operatorname{int} A \neq \varnothing$.

The set A is said to be *convex* if for any points x' and x'' in A the line segment joining them, $tx' + (1-t)x''$, $0 \leqslant t \leqslant 1$, lies entirely in A. We shall use $\operatorname{ch} A$ to denote the *convex hull* of a set A.

A *cone* in $\mathbf{R}^n$ (with the vertex at 0) is a set Γ with the property that if $x \in \Gamma$, then λx also belongs to Γ for all $\lambda > 0$. Denote by $\operatorname{pr} \Gamma$ the intersection of Γ with the unit sphere S^{n-1} (Fig. 2). A cone Γ' is said to be *compact* in the cone Γ if $\operatorname{pr} \Gamma' \subset\subset \operatorname{pr} \Gamma$ (fig. 2); then we write $\Gamma' \subset\subset \Gamma$.

The cone

$$\Gamma^* = [\xi : (\xi, x) \geqslant 0, x \in \Gamma]$$

is said to be *dual* to the cone Γ (Fig. 2). Clearly, Γ^* is a closed convex cone with its vertex at 0 and $(\Gamma^*)^* = \overline{\operatorname{ch} \Gamma}$. A cone Γ is said to be *acute* if there exists a supporting plane for $\overline{\operatorname{ch} \Gamma}$ that has a unique common point 0 with $\overline{\operatorname{ch} \Gamma}$ (Fig. 2). This condition is equivalent to the condition of the cone Γ^* being solid (see Section 1.9). We write $\operatorname{int} \Gamma^* = C$.

Let Γ be an acute cone in $\mathbf{R}^n$ and $C = \operatorname{int} \Gamma^*$. A smooth $(n-1)$-dimensional surface S without a border is said to be C*-like* if for any $x \in S$ the cone $\overline{\operatorname{ch} \Gamma} + x$ intersects S at the single point x. We denote by S_+ and S_- those (open) parts of $\mathbf{R}^n$ which contain the cones $\overline{\operatorname{ch} \Gamma} + x$, $x \in S$, and $\overline{\operatorname{ch} \Gamma} - x$, $x \in S$, respectively, so that $S_+ \bigcup S \bigcup S_- = \mathbf{R}^n$ (Fig. 3).

Examples of convex acute cones. (1) The *positive octant* in $\mathbf{R}^n$:

$$\mathbf{R}_+^n = [x : x_1 > 0, \ldots, x_n > 0], \ \left(\mathbf{R}_+^n\right)^* = \bar{\mathbf{R}}_+^n.$$

(2) The *future light cone* in $\mathbf{R}^{n+1}$:

$$V_n^+ = [x = (x_0, \mathbf{x}) : x_0 > |\mathbf{x}|], \ \left(V_n^+\right)^* = \bar{V}_n^+.$$

Here $\mathbf{x} = (x_1, \ldots, x_n)$.

(3) The *origin of coordinates* $\{0\}$; $\{0\}^* = \mathbf{R}^n$.

Let A be a set in $\mathbf{R}_y^n$. The *tube set* $\mathbf{R}^n + iA = [z = x + iy : x \in \mathbf{R}^n, y \in A]$ in C^n is denoted by T^A.

0.3. The *Lebesgue integral* of a (complex-valued, measurable) function $f(x)$ over $\mathbf{R}^n$ is denoted by

$$\int_{\mathbf{R}^n} f(x)\, dx_1 \cdots dx_n = \int f(x)\, dx;$$

and over a measurable set A by

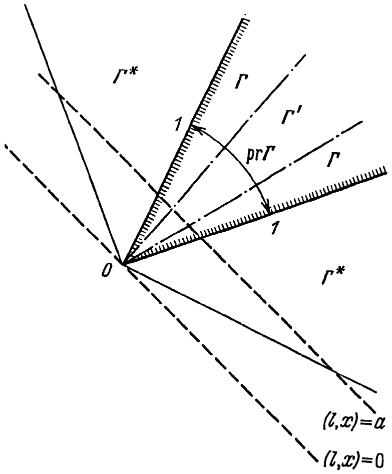

Figure 2.

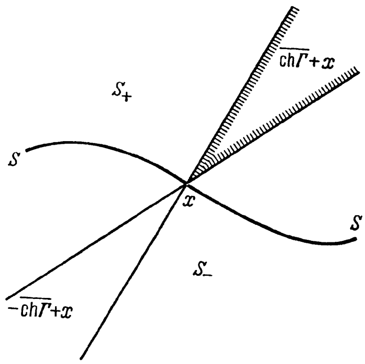

Figure 3.

$$\int_A f(x)\,dx = \int \theta_A(x) f(x)\,dx.$$

The set of all functions f for which the norm

$$\|f\|_{\mathcal{L}^p(A)} = \begin{cases} \left[\int_A |f(x)|^p\,dx\right]^{1/p}, & 1 \leqslant p < \infty; \\ \operatorname*{ess\,sup}_{x \in A} |f(x)|, & p = \infty. \end{cases}$$

is finite, will be denoted by $\mathcal{L}^p(A)$; we write $\|f\| = \|f\|_{\mathcal{L}^2(\mathbf{R}^n)}$, $\mathcal{L}^p(\mathbf{R}^n) = \mathcal{L}^p$.

If $f \in \mathcal{L}^p(A')$ for every $A' \subset\subset A$, then f is said to be *p-locally summable* in A (for $p = 1$ we say that it is *locally summable* in A). The set of all functions p-locally summable in A is denoted by $\mathcal{L}^p_{loc}(A)$; $\mathcal{L}^p_{loc}(\mathbf{R}^n) = \mathcal{L}^p_{loc}$.

A measure $d\mu(x)$ defined on a Borel set $A \subset \mathbf{R}^n$ is said to be *locally finite* in A if for every compact $K \subset\subset A$, $\int_K |d\mu(x)| < \infty$; and is said to be *finite* on A if

$$\int_A |d\mu(x)| < \infty.$$

A measure $d\mu(x)$ (or a function $f(x)$), defined on an unbounded Borel set $A \subset \mathbf{R}^n$, is said to be *tempered* on A if for some $m \geqslant 0$

$$\int_A (1+|x|)^{-m}\,|d\mu(x)| < \infty \quad (\text{resp.} \int_A (1+|x|)^{-m}\,|f(x)|\,dx < \infty)$$

0.4. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a *multi-index;* that is, the components α_j are nonnegative integers, $\alpha \in \mathbf{Z}_+^n$. We use the following notation:

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

$$\begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \begin{pmatrix}\beta_1\\ \alpha_1\end{pmatrix}\begin{pmatrix}\beta_2\\ \alpha_2\end{pmatrix} \cdots \begin{pmatrix}\beta_n\\ \alpha_n\end{pmatrix} = \frac{\alpha!}{\beta!(\alpha-\beta)!},$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n .$$

Let $\partial = (\partial_1, \partial_2, \ldots, \partial_n)$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, \ldots, n$. Then

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|} f(x_1, x_2, \ldots, x_n)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} = \partial_1^{\alpha_1}\partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} f(x).$$

We write $\alpha \geqslant \beta$ iff $\alpha_j \geqslant \beta_j$ $j = 1, \ldots, n$. Sometimes we shall use the notation $\alpha = (\alpha_1, \ldots, \alpha_n)$ for a multi-index with components α_j of any sign, $\alpha \in \mathbf{Z}^n$.

Let $\mathcal{O}$ be an open set in $\mathbf{R}^n$. We use $C^k(\mathcal{O})$ to denote the set of all functions $f(x)$ which are continuous in $\mathcal{O}$ together with all derivatives $\partial^\alpha f(x)$, $|\alpha| \leqslant k$; $C^\infty(\mathcal{O})$ is the set of all infinitely differentiable functions in $\mathcal{O}$. The set of all functions $f(x)$ from $C^k(\mathcal{O})$, $0 \leqslant k \leqslant \infty$, for which all derivatives $\partial^\alpha f(x)$, $|\alpha| \leqslant k$, admit continuous extension on $\bar{\mathcal{O}}$ will be denoted by $C^k(\bar{\mathcal{O}})$. We introduce the norm $C^k(\bar{\mathcal{O}})$, $0 \leqslant k \leqslant \infty$, by the formula

$$\|f\|_{C^k(\bar{\mathcal{O}})} = \sup_{x\in\mathcal{O},\, |\alpha| \leqslant k} |\partial^\alpha f(x)|.$$

We also denote by $C_0^k(\bar{\mathcal{O}})$, $0 \leqslant k \leqslant \infty$, the set of all functions from $C^k(\bar{\mathcal{O}})$ which vanish on $\partial\mathcal{O}$ together with all their derivatives up to the order k.

The *support* of a function $f(x)$ continuous in $\mathcal{O}$ is the closure in $\mathcal{O}$ of the set of those points for which $f(x) \neq 0$; the support of f is denoted by $\operatorname{supp} f$; that is

$$\operatorname{supp} f = [\overline{x : f(x) \neq 0, x \in \mathcal{O}}].$$

If $\operatorname{supp} f \subset\subset \mathcal{O}$ then f is called *finite* in $\mathcal{O}$.

We denote by $C_0^k(\mathcal{O})$ the set of all functions of the class $C^k(\mathcal{O})$, which are finite in $\mathcal{O}$, $0 \leqslant k \leqslant \infty$. We write $C(\mathcal{O}) = C^0(\mathcal{O})$, $C(\bar{\mathcal{O}}) = C^0(\bar{\mathcal{O}})$, $C_0(\mathcal{O}) = C_0^0(\mathcal{O})$, $C_0(\bar{\mathcal{O}}) = C_0^0(\bar{\mathcal{O}})$, $C^k = C^k(\mathbf{R}^n)$.

We shall say that a vector function (or a matrix function) possesses some property if each of its components possesses this property.

A map $x \to y(x) = (y_1(x), \ldots, y_n(x))$ from an open set $\mathcal{O} \subset \mathbf{R}_x^n$ onto an open set $\mathcal{O}_1 \subset \mathbf{R}_y^n$ is called a C^k*-diffeomorphism* (or C^k*-smooth diffeomorphism),* $0 \leqslant k \leqslant \infty$, if $y \in C^k(\mathcal{O})$, it is one-to-one, and the inverse map $y \to x(y)$ belongs to $C^k(\mathcal{O}_1)$; an C^0-diffeomorphism is called a *homeomorphism.*

For C^k-diffeomorphisms, $k \geqslant 1$, the Jacobian is nonzero:

$$J\left(\frac{y}{x}\right) = \det\left(\frac{\partial y_i(x)}{\partial x_j}\right) \neq 0, \ x \in \mathcal{O}$$

0.5. *Miscellaneous notation.*

$$\sigma_n = \int_{S^{n-1}} d\sigma = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

is the surface area of the unit sphere S^{n-1}; $d\sigma$ is the Lebesgue measure on the sphere S^{n-1}.

The transpose of a matrix $A = (a_{kj})$ is denoted by $A^T = (a_{jk})$. We denote by I the unit matrix, and by g a diagonal with elements $[1, -1, \ldots, -1]$ on the diagonal. For every $n \times n$ real matrix A we associate the linear transformation $x \to Ax$ of the space $\mathbf{R}^n$.

We denote by $O(n)$ the *rotation group* of $\mathbf{R}^n$ (the set of all real linear transformations $x \to Ax$ such that $A^T = A^{-1}$; $SO(n)$ is the *proper rotation group* of $\mathbf{R}^n$ ($A \in O(n)$, $\det A = 1$); $L_+^\uparrow(n)$ is the *proper orthochronous Lorentz group* of $\mathbf{R}^{n+1}$ ($A^T gA = g$, $\det A = 1$, $a_{00} > 0$); and $T(n)$ denotes the *translation group* of $\mathbf{R}^n$ ($x \to x + b$, $b \in \mathbf{R}^n$).

We denote the *uniform convergence* of a sequence of functions $\{\phi_k(x), k \to \infty\}$ to a function $\phi(x)$ on a set A by:

$$\phi_k(x) \overset{x \in A}{\Longrightarrow} \phi(x), \ k \to \infty;$$

if $A = \mathbf{R}^n$, then instead of $\overset{x \in \mathbf{R}^n}{\Longrightarrow}$ we shall write $\overset{x}{\Longrightarrow}$.

We denote the convergence of a sequence of elements $\{f_k, k \to \infty\}$ to an element f in a topological vector space B by: $f_k \to f$, $k \to \infty$ in B.

We denote by $\omega_\epsilon(x)$ the *normed bell-shaped function* (Fig. 4)

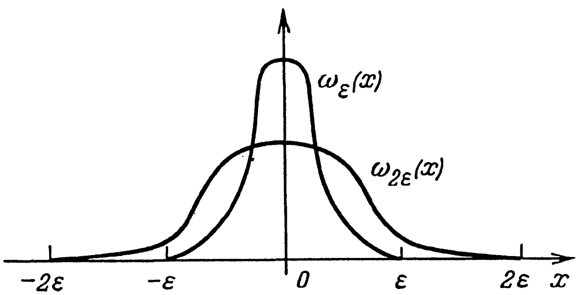

Figure 4.

$$\omega_\epsilon(x) = \begin{cases} C_\epsilon \exp\left(-\dfrac{\epsilon^2}{\epsilon^2 - |x|^2}\right), & |x| \leqslant \epsilon, \\ 0 & , \ |x| > \epsilon, \quad \int \omega_\epsilon(x)\,dx = 1. \end{cases}$$

Let α be a real number; we denote by $[\alpha]$ the *greatest integer* which is not larger than α.

The *wave operator (d'Alembertian)* is denoted by

$$\Box_n = \partial_0^2 - \partial_1^2 - \cdots - \partial_n^2.$$

Let A and B be two arbitrary sets. We shall denote by $A \times B$ the set of all pairs (a, b), $a \in A$, $b \in B$.

The sections are numbered in a single sequence. Each section consists of subsections; references to a subsection contain the number of the corresponding section. Formulae are numbered separately in each subsection, and they also contain the number of the subsection. When a reference is made to a formula in another section, the number of that section is also given.