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Questions of Uniqueness and Resolution in Reconstruction from Projections

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In memory of my father, Julius.

This is one of his accomplishments.

PREFACE

Reconstruction from projections has revolutionized radiology and as now become one of the most important tools of medical diagnosis - he E. M. I. Scanner is one example. In this text, some fundamental heoretical and practical questions are resolved.

Despite recent research activity in the area, the crucial subject f the uniqueness of the reconstruction and the effect of noise in the ata posed some unsettled fundamental questions. In particular, Kennan mith proved that if we describe an object by a C_0^∞ function, i.e., nfinitely differentiable with compact support, then there are other bjects with the same shape, i.e., support, which can differ almost rbitrarily and still have the same projections in finitely many direc- ions. On the other hand, he proved that objects in finite dimensional unction spaces are uniquely determined by a single projection for almost ll angles, i.e., except on a set of measure zero. Along these lines, erman and Rowland in [41] showed that reconstructions obtained from he commonly used algorithms can grossly misrepresent the object and hat the algorithm which produced the best reconstruction when using oiseless data gave unsatisfactory results with noisy data. Equally mportant are reports in Science, [67, 68] and personal communications y radiologists indicating that in medical practice failure rates of econstruction vary from four to twenty percent.

Within this work, the mathematical dilemma posed by Kennan Smith's esult is discussed and clarified. A larger class of function spaces is hown to have a certain pathological property which maintains that any inite number of projections is not sufficient to specify the desired bject function. As this property is not shared by finite dimensional unction spaces, one is led to the study of discretized models even in he most abstract of approaches.

The main thrust of this work is concerned with the limitations on he application of reconstruction from projections imposed by practical iagnostic medical techniques. The particular finite dimensional function pace examined is exactly the one most often used in brain scanning evices. Attention is paid to the empirical tradeoff between resolution nd accuracy in projection data. Paramount, however, is the goal of ptimal resolution in the reconstruction. This has clear diagnostic nd theoretical significance since many medical anomalies require the inest image detail for their unambiguous identification. High resolu- ion is also important to the justification of the finite dimensional econstruction as a good approximation to the actual object. The medical ontext of this report is clarified in an appendix.

Two independent results indicate that a certain set of projection angles should be used. Theorem 1, in Chapter V, shows that this set of angles is the least demanding on the resolution of the projection data - thereby establishing that projection data with the highest stati tical accuracy can be utilized at these angles. Theorem 2, in the sixt chapter, provides a simple formula specifying the resolution in the reconstruction which cannot be exceeded if (uniquely) determined information is required. That is, the extent to which an object is determi by its projection data depends on how one defines the approximation, i.e., the resolution of the reconstruction. The applicability of Theo 2 requires that the projection angles come from the previously descri set.

In Chapter VIII, it is shown that once physically justifiable assu tions are made, it is possible to predict how well a reconstruction approximates the original object function. An estimate which indicates the accuracy of the reconstruction is given.

The style of this work was chosen so that researchers in the general field of image reconstruction as well as interested physicians would find the ideas presented accessible. Although the mathematical treatment is not self-contained, nothing beyond second or third year college mathematics is assumed. In particular, the ideas developed in a course in linear algebra (or vector spaces) and the concept of a norm on a vector space are considered common or easily accessible knowledge. To facilitate reading, proofs of propositions and theorems are deferred until the ends of the chapters in which they occur.

This manuscript owes much to the help of many others. The origina suggestion came from Hans J. Bremermann who followed up that suggestion with valuable insight and encouragement. Frequent consultation with Ole Hald, Paul Chernoff, William Bade, Kennan T. Smith, Alberto Grunbau and Keith Miller provided support and infectious enthusiasm. Special efforts were made by Jaime Milstein, Kim Chambers and Nora Lee. I extend many thanks to David Krumme for his conscientious and indefatigable work in editing and criticizing every draft, and to Cherie Parker, for sustained moral support during a difficult period.

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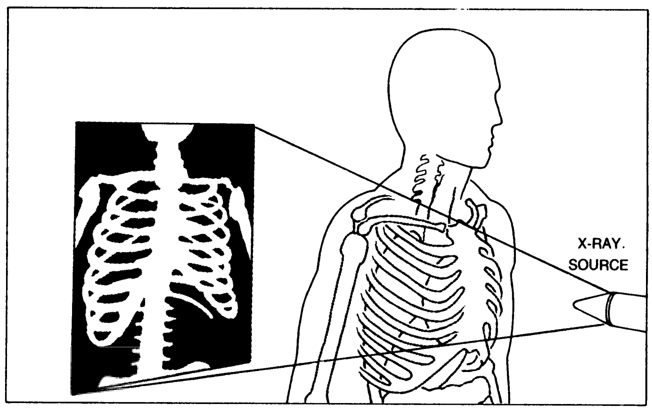
CHAPTER I

DESCRIPTION OF THE GENERAL PHYSICAL PROBLEM

Consider an object with varying density (of mass). For example: A human body has bones, lungs, muscles and etc. It is possible to make the "inner structure" visible through the use of x-rays, but we only get a projection. How could the actual three dimensional mass distribution be determined?

Figure I.1
[34]



CONVENTIONAL X-RAY PICTURE is made by allowing the X rays to diverge from a source, pass through the body of the subject and then fall on a sheet of photographic film.

This report attacks the general problem of reconstruction from projections. That is, the problem of determining the inner structure of an object from a finite number of projections of that object. In addition, the exposition is concerned with the practical problem of reconstruction from projections, which is subject to inaccurate data collection and a limited amount of information in each projection. The problem of determining the mass distribution within a human body with x-ray projection data is important to medical diagnosis. It is used as a typical example of a more general class of reconstruction-from-projections problems.

From the outset this report considers all problems to be two-dimensional. A three-dimensional problem can be reduced to a two-dimensional problem by recognizing that if a horizontal (planar) cross-section of the mass distribution is known at every height then all of the three-dimensional information is readily available. We are now considering projection data to be the information within the radiograph along a line which is contained in the cross-sectional plane of interest. By this change in the definition of the projection data, we can convert a three-dimensional problem into a two-dimensional problem.

When x-rays pass through matter, they are attenuated roughly in proportion to the density of that matter. Thus if we record the intensity of x-rays that have passed through an object, we get a projection of the mass density. If the purpose of such an experiment is to determine a cross-section of the density distribution in a man's head, then the physical problem is reduced to recovering that distribution from a finite number of radiographs.

In the beginning, it is well to translate this problem into a clear mathematical question. We describe the given mass distribution by its density at each point. Let f be a real-valued function defined on a plane. (f actually represents the coefficient of linear x-ray attenuation at each point within the particular cross-sectional plane.)

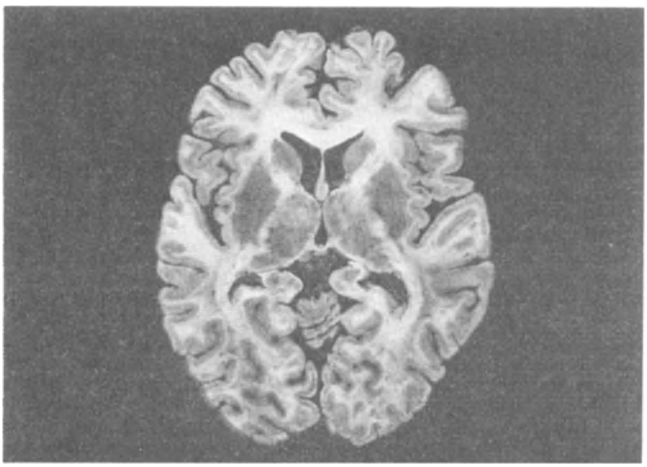


Figure I.2 [34]
A photograph of a cross-section of a dissected normal brain.

Because f represents a physical object, it is a non-negative function which is considered to be zero in air (since attenuation of x-rays by air is small). f will be called the *objective function*, inasmuch as recovering f is the goal of the analysis. In particular, this means that f is only non-zero on a bounded subset of the plane, i.e., f has compact support. By rescaling the plane, if necessary, another simplifying assumption can be made: f is non-zero only within the unit square I^2, i.e., $f(x,y) \neq 0$ implies that $0 \leq x,y \leq 1$.

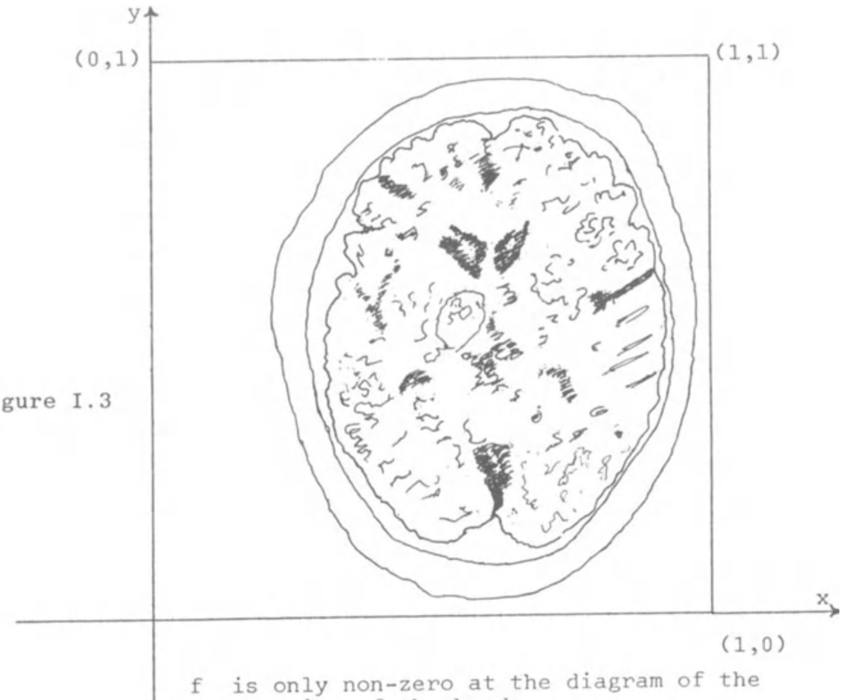


Figure I.3

f is only non-zero at the diagram of the cross-section of the head.

Once we have chosen a coordinate system a *projection* is characterized by an *angle*, θ , which we measure in a counter-clockwise direction from the x-axis.

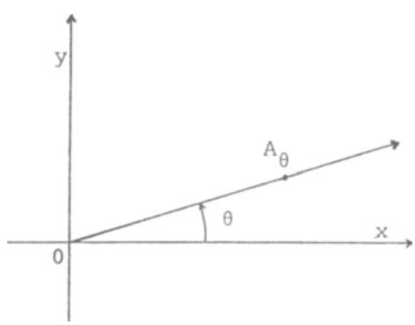
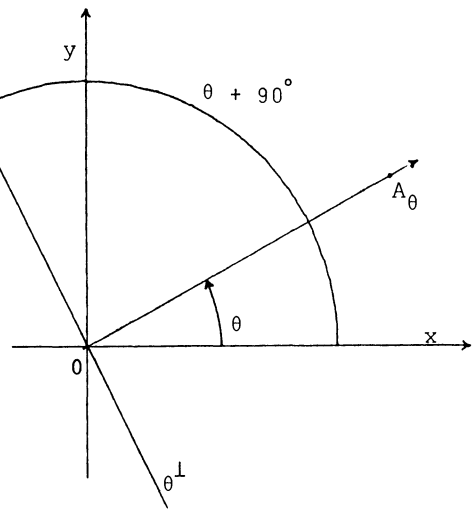


Figure I.4

Let A_θ be any point on the ray determined by θ . The ray, $\overrightarrow{OA_\theta}$, is completely determined by θ and in turn specifies $\theta^{\perp}$, the line perpendicular to $\overrightarrow{OA_\theta}$ and passing through 0 .

Figure I.5



A *projection of* f *in the direction* θ is given by:

<u>I.1</u> $$P_\theta f(v) = \int_{-\infty}^{+\infty} f(v + tA_\theta)dt \text{ , for } \mathbf{v} \text{ in } \theta^{\perp} .$$

That is, $P_\theta f$ is a new function which is defined on the line $\theta^{\perp}$ and has the property that its value at any point v in $\theta^{\perp}$ only depends on that part of f defined on the line $\{v + tA_\theta : t \text{ is in } R\}$.

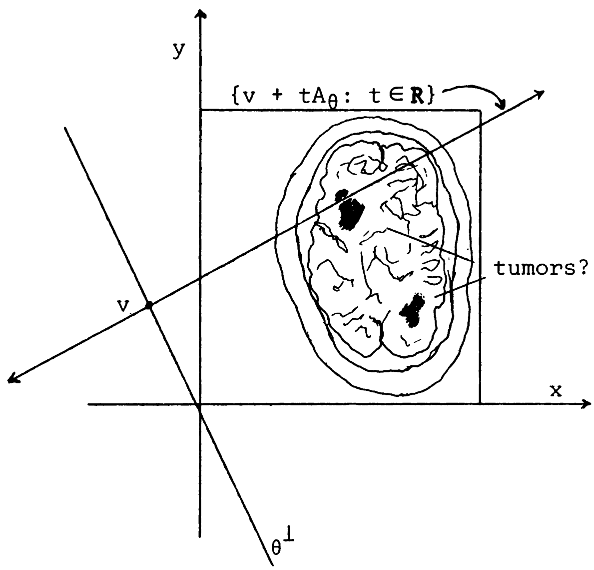


Figure I.6

When f represents the distribution of x-ray attenuation in a cross-section of a human head, then $P_\theta f$ is usually just an x-ray or radiograph of that head read along the appropriate line on the photographic film. (In actual practice the optical density measured records $\exp(P_\theta f(v))$ so that the original data must be converted with the logarithm to get the desired projection information. See pages 155-157.)

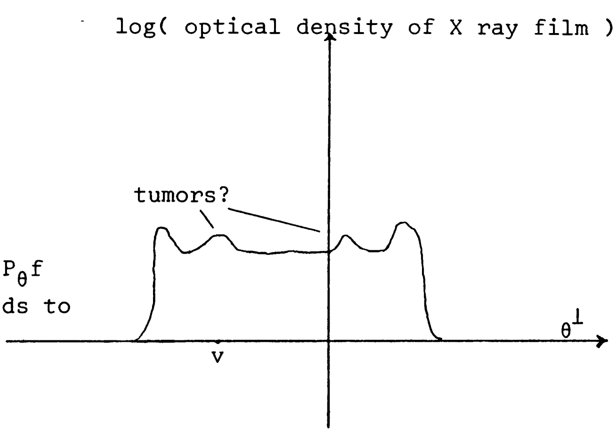


Figure I.7

Plot of $P_\theta f$
(v corresponds to Figure I.6)

For notational convenience, it can be assumed that A_θ has length one, so that A_θ is uniquely determined by θ. It is reasonable to drop the name A_θ in favor of $\vec{\theta}$. This should cause no confusion since the meanings of θ and $\vec{\theta}$ are naturally related:

(1) θ is an *angle* measured in the counter-clockwise direction from the x-axis, and (2) $\vec{\theta}$ is the *vector* with length one which makes the same angle with the x-axis.

With this convention, the fundamental definition of a *projection of* f *in the direction* $\vec{\theta}$ becomes:

$$\underline{\text{I.2}} \qquad P_\theta f(v) = \int_{-\infty}^{+\infty} f(v + t\vec{\theta})\, dt, \quad \text{for } v \text{ in } \theta^\perp .$$

In the present introductory discussion a large class of mathematical questions like the well-definedness of P_θ is postponed, but the rigorous reader should be satisfied with the single condition that f be piecewise continuous.

Because of time and numerous other considerations, any physical measurement only contains a finite amount of information. The time limitation translates into very significant economic constraints: the present imaging techniques cost about $200 to the patient. Furthermore, some organs, like the breast, are very sensitive to radiation damage. Because of the need to protect the patient from an excessive dose and because of monetary factors, it is desirable to take as *few* projections as possible as is consistent with the *goal* to get a medically acceptable reconstruction.

For notational convenience, the letter m will be used to indicate the number of projection angles used, i.e., the observed data is $\{P_{\theta_1}f, P_{\theta_2}f, \ldots, P_{\theta_m}f\}$, corresponding to the projection angles $\{\theta_1, \theta_2, \ldots, \theta_m\}$. The general mathematical question is to determine f from $\{P_{\theta_1}f, P_{\theta_2}f, \ldots, P_{\theta_m}f\}$.

The EMI Scanner - An Example of the Present State of the Art

Determining the coefficient of x-ray absorption everywhere within a cross-section of a human head from a finite number of radiographs is an example of the type of problem addressed in this report. This problem is particularly difficult because variations of density are small. It is an important problem since the solution would make it possible to detect brain tumors. There are many other physical problems which can be modeled with the same mathematical machinery. Examples of these problems range from radio astronomy to plasma physics, and from study of air flow in a wind tunnel to electron microscopy. (See later discussion about determining the structure of protein molecules with projection data obtained from an electron microscope, p.14.) Medical practitioners have become interested in reconstruction from projections surpassing its long standing use in nuclear medicine: a new device called the EMI Scanner has proven to be generally reliable in producing mass density distributions of cross-sections of human brains from x-ray projection data. So that at least one example is developed in full, the EMI Scanner is explained.

The EMI Scanner processes x-ray projection data and produces a reconstruction (i.e., finite dimensional approximation) of the actual mass distribution of a cross-section of the patient's head. The first problem to be overcome is the method of data collection. This can be accomplished by scanning the patient and feeding this data into a computer as in Figure I.8.

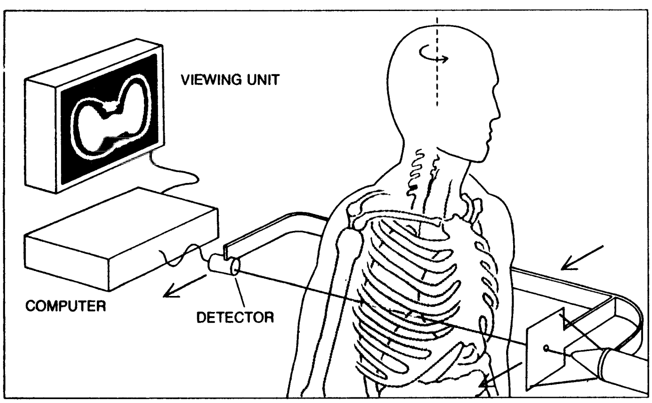
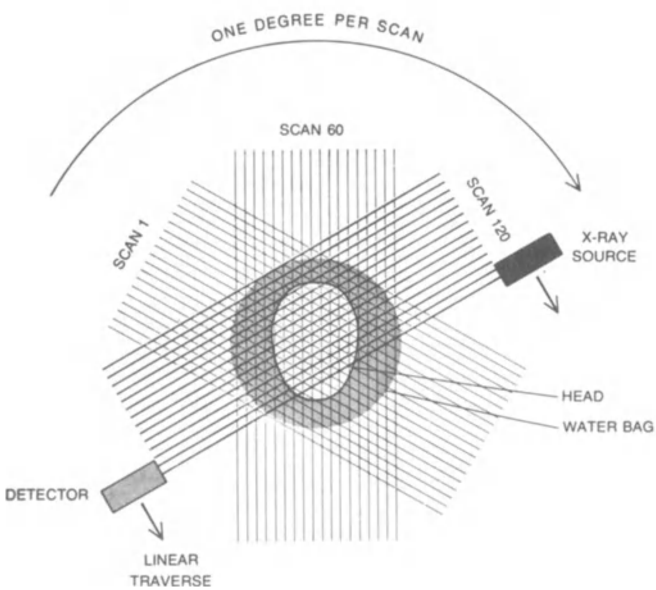


Figure I.8
[34]

RECONSTRUCTION FROM PROJECTIONS is made by mounting the X-ray source and an X-ray detector on a yoke and moving them past the body. The yoke is also rotated through a series of angles around the body. Data recorded by detector are processed by a special computer algorithm, or program. Computer generates picture on a cathode-ray screen.

In the particular case of the EMI, the scanning is restricted only to the head. The next photograph shows a patient lying on a table so that her head can be sampled with the scanning equipment of the EMI.



SCANNER SAMPLES RAY SUMS AT 160 POINTS along each projection; in five and a half minutes 180 projections are taken at one-degree intervals around the patient's head.

Figure I.9 [1] [34]

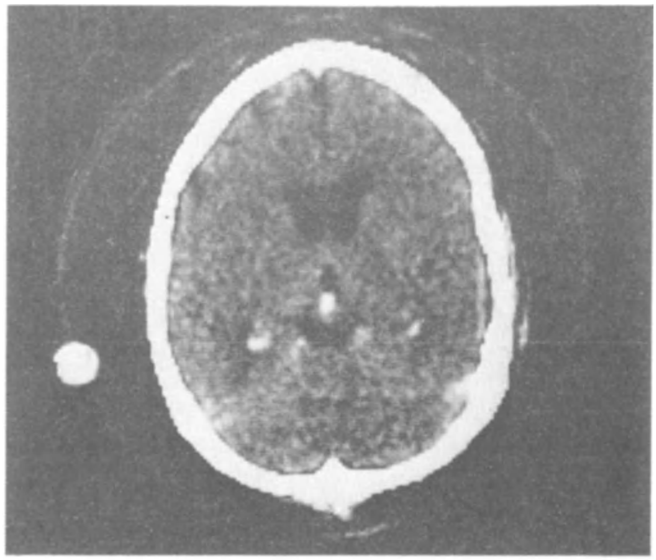
In this case, the head is held in place by a water bag. Then a scanning detector system takes a projection at each of 180 different angles in 1° increments. Each projection is actually 160 different measurements so that 160 × 180 observations are inputed into a computer for processing.

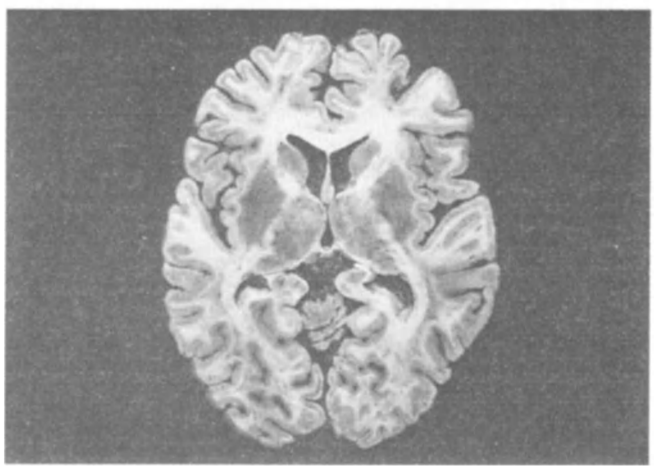
The x-ray equipment will have inputed 11,200 (160×180) different numbers which represents $\{P_{\theta_1} f, P_{\theta_2} f, \ldots, P_{\theta_{180}} f\}$ where $\theta_1 = 1°, \theta_2 = 2°, \ldots, \theta_{90} = 90°, \ldots, \theta_{180} = 180°$. In this case, however, each projection has been divided into 160 different data collection regions.

The next step in the process is the use of a computer implemented algorithm which transforms the projection data into the desired reconstruction. For some time, the algorithm called the Algebraic Reconstruction Technique or *ART* developed by Herman, Gordon, and Bender [32], was used to perform the desired transformation. Later the *convolution* method of Ramachandran and Lakshminarayanan [63], was implemented. [41] (See pp 162-168 for more explanation.)

Finally, the reconstruction is displayed on a cathode ray tube or T.V. screen. Figure I.10 compares a reconstruction with an actual cross-section; notice that the obtained reconstruction is far less complex than the part of the brain that it is supposed to represent.

Figure I.10 [34]





COMPARISON OF TWO CROSS SECTIONS, one an image of a normal human brain obtained by the EMI scanner (*top*) and the other a photograph of a section of a dissected normal human brain (*bottom*), illustrates the operation of the scanner. The anatomical details of the two brains can be readily compared. The reconstructed spot that appears to the left of the scanner image corresponds to a plastic rod that is used for calibrating the X-ray densities in the image. The scanner image of the head consists of 11,200 (160 × 160) picture elements. The image was made in the course of a study by D. F. Reese of the Mayo Clinic.