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Computational Tensor Analysis of Shell Structures

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PREFACE

This book presents a method which is capable of evaluating the deformation characteristics of thin shell structures A free vibration analysis is chosen as a convenient means of studying the displacement behaviour of the shell, enabling it to deform naturally without imposing any particular loading conditions. The strain-displacement equations for thin shells of arbitrary geometry are developed. These relationships are expressed in general curvilinear coordinates and are formulated entirely in the framework of tensor calculus. The resulting theory is not restricted to shell structures characterized by any particular geometric form, loading or boundary conditions.

The complete displacement and strain equations developed by Flugge are approximated by the curvilinear finite difference method and are applied to computing the natural frequencies and mode shapes of general thin shells. This approach enables both the displacement components and geometric properties of the shell to be approximated numerically and accurately.

The selection of an appropriate displacement field to approximate the deformation of the shell within each finite difference mesh is discussed in detail. In addition, comparisons are made between the use of second and third-order finite difference interpolation meshes.

The method is applied to a series of flat plate, cylindrical, spherical and conical shell structures to verify its accuracy. The computed natural frequencies and their associated mode shapes are documented and compared with published results.

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NOTATION

General

[ ]	Matrix notation.
{ }	Vector notation.
$( )_{,\alpha}$	Partial differentiation with respect to the surface coordinates.
$( )\vert_{\alpha}$	Covariant differentiation with respect to the coordinates x^i.
$( )\Vert_{\alpha}$	Covariant differentiation with respect to the surface coordinates x^α.
$(\dot{\ })$	Time derivative.
$[\ ]^t$	Transpose of a matrix.
$[\ ]^{-1}$	Inverse of a matrix.
[K]	Structure's stiffness matrix
[M]	Structure's mass matrix.
$\mathbb{R}$	Two dimensional region.
$\mathbb{S}$	Boundary curve.
r_i	Integration point for a Gauss-Legendre numerical integration scheme.
w_i	Weight factors for a Gauss-Legendre numerical integration scheme.
x^i	Coordinates of a general three-dimensional curvilinear coordinate system.
x^α	Coordinates of a general two-dimensional curvilinear coordinate system on the shell's middle surface.
x^3	Coordinate axis normal to x^α.
δ_i^j	Kronecker delta.
λ	Eigenvalue.

ψ	General two-dimensional scalar function.
Ψ_i	Scalar quantities at nodal points.

Material Properties

K	Bending stiffness: $Et^3/(12(1-\nu^2)$.
C	Damping constant.
E	Young's modulus of elasticity.
t	Shell's thickness.
ν	Poisson's ratio.
ρ	Material density.

Geometric Properties

$\mathbf{a}_\alpha$, $\mathbf{a}^\alpha$	Covariant and contravariant base vectors at a point x^α on the surface z = 0.
$\mathbf{a}_3$, $\mathbf{a}^3$	Covariant and contravariant unit normal base vectors.
$a_{\alpha\beta}$, $a^{\alpha\beta}$	Covariant and contravariant components of the metric tensor on the surface z = 0.
$b_{\alpha\beta}$, $b^{\alpha\beta}$, b^β_α	Covariant, contravariant and mixed components of the curvature tensor.
dA	Area element.
ds	Line element.
dV	Volume element.
$\mathbf{g}_i$, $\mathbf{g}^i$	Covariant and contravariant base vectors at a point x^α on the surface $z \neq 0$.

g_{ij} g^{ij}	Covariant and contravariant components of the metric tensors at a point x^{α} on the surface $z \neq 0$.
g	determinant of the metric tensor component matrix.
i, j, k	Base vectors in the cartesian coordinate system.
[J]	Jacobian matrix.
n, t, s	Base vectors associated with the boundary coordinate system.
r	Position vector of a point on the surface $z \neq 0$ before deformation.
s	Position vector of a point on the surface $z=0$ before deformation.
x, y, z	Cartesian coordinates.
z	A coordinate along the x^3 axis.
β_i^j	Transformation coefficients.
$\in_{\alpha\beta}$, $\in^{\alpha\beta}$	permutation tensors.
Γ_{ij}^k Γ_{ijk}	Christoffel symbols.
μ_{α}^{β}, λ_{α}^{β}	Shift tensors.

Deformation

f	A vector defining the extension and rotation of the middle surface resulting from relaxing the conservation of normals assumption.
u	Middle surface displacement vector.
v	General surface displacement vector.
w	Displacement component in the direction normal to the shell surface.
$\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$, $\varepsilon_{\alpha}^{\beta}$	Covariant, contravariant and mixed components of the middle surface in-plane strain tensor.

$\eta_{\alpha\beta}, \eta^{\alpha\beta}, \eta_{\alpha}^{\beta}$	Covariant, contravariant and mixed components of the general surface in-plane strain tensor.
γ_{ij}	General three-dimensional strain tensor.
$\kappa_{\alpha\beta}$ $\kappa^{\alpha\beta}$	Covariant and contravariant components of the change in curvature tensor.

Stresses

$\sigma^{\alpha\beta}, \sigma_{\alpha}^{\beta}$	Contravariant and mixed components of the stress tensor.
$M^{\alpha\beta}$	Contravariant components of the bending stress tensor.
$N^{\alpha\beta}$	Contravariant components of the membrane stress tensor.
$\{P\}$	External force vector.

1. INTRODUCTION

Shell structures are widely used in a variety of engineering applications ranging from domes for major buildings and components of flight structures to liquid storage containers. These structures often have arbitrary shape and support conditions to meet functional and manufacturing requirements.

In modelling a shell numerically, an attempt is made to summarize its three dimensional behaviour in terms of the deformation of its middle surface. In order to achieve this, a number of approximations must be introduced into the formulation. Unlike the theory of plates where the differential equation of motion is universally agreed upon, the introduction of various geometric and displacement approximations give rise to a large number of shell theories.

The works of Flugge [1], Donnell [2], Love [3], Mushtari [4], Naghdi [5], Timoshenko [6], Reissner [7] and Vlasov [8] apply to shells of arbitrary curvature and are based on Love's 'first approximation' [3]. Essentially, differences between theories arise from simplifications involving the thickness to radius of curvature terms.

Comparisons of the various thin shell theories are documented in the works of Leissa [9] and Kraus [10]. A brief discussion on their suitability to modern computerized structural analysis is given by Bushnell [11].

Historically the thin shell equations of motion were first applied to structures characterized by a prescribed analytical geometry and constrained by ideal boundary conditions. This enabled the governing differential equations to be simplified and analytical solutions to be obtained for the dynamic characteristics of the structure. References [12] - [20] are typical of the approach used by early researchers. In addition, many authors (e.g. references [21] - [27]) have combined various numerical techniques with simplified versions of the shells governing strain-displacement equations. The literature in this area is extensive. Leissa [9] in his monograph 'Vibration of Shells' presents a comprehensive survey and discussion of the vibration of cylindrical, spherical and conical based shells. Since its release in 1973 new formulations, particularly in the area of vibrating shell panels, have been developed [24] - [27].

The requirement to analyse structures of arbitrary shape with irregular loading and support conditions limits the scope of the available analytical solutions. One technique which has been used successfully for the analysis of general shell structures since its introduction in the early 1960's is the Finite Element Method.

The literature detailing the application of the method to the dynamic analysis of shell structures is extensive. Only a brief overview is given here and references cited are indicative of the work carried out in the field. A detailed list of references has been compiled by Norrie and deVries [28].

One of the first applications to appear in the literature was the free vibration of shells of revolution [29] - [41]. Theoretically, the geometric nature of the problem allows the complex strain-displacement equations to be simplified. In early work [30], the shell was represented by a stacked assembly of elements which geometrically duplicated the frustum of a cone. Each element was permitted to deform in membrane and bending states and the continuity of displacement and slope was enforced on nodal circles along which neighbouring elements are connected. Continuing refinement in element technology has seen the introduction of general curved elements to improve the geometric representation of the surface[36].

The development of the flat plate bending element enabled the curved shell surface to be replaced by an assembly of flat triangular or rectangular elements [42] - [46]. The elements are formed by the superposition of stretching and bending behaviour which are uncoupled provided deformations are small. Many of the flat plate elements [48] differ from each other by the choice of shape functions and by the connections imposed between adjacent elements.

A number of difficulties and shortcomings of the methodology have been discussed in the literature [49,50]. These include the presence of discontinuity moments at the junction of adjacent elements and the difficulty of treating the junction where all elements are coplanar. As noted by Zienkiewicz [46] and Gallagher [50], the discontinuity moments can in many cases be suppressed by using a large number of elements.

The need to represent the surface geometry accurately has led to the development of a number of curved thin shell elements [51] - [56]. Many elements (e.g. Olsen & Lindberg [52]) are formulated for specific geometric configurations thus eliminating complex nodal parameter transformations.

The use of 'shallow shell theory' has enabled the development of a number of shallow curved elements [53, 54]. In particular, Olsen and Lindberg [54] reported the results of a high order conforming element which utilized first and second order derivatives at each corner of the element. Since the resultant nodal derivative parameters include only those derivatives with respect to the x-y coordinate system, the formulation is restricted to analysing shell structures which have the correct geometric properties and can be orientated in space to account for the implementation's shallow shell assumptions.

Zienkiewicz et al [57] presented an isoparametric shell element based on three dimensional theory. This element, formed by degenerating the three dimensional solid isoparametric element, was used successfully for a variety of relatively thick shell problems. As reported by Gill and Ucmaklioglu [58], the element was found to be over stiff when applied to thin shell situations. A modified version of the element using a reduced integration technique for the determination of the element stiffness matrix was applied to vibration problems by Hofmeister and Evensen [59] and by Bossak and Zienkiewicz [60]. The technique was found to produce acceptable results for the analysis of flat, spherical and cylindrical shells. However, the element performed poorly when applied to the case of a turbine blade. The inaccurate results were attributed to the failure of the reduced integration scheme to capture the rapid local change in curvature along part of the blade.

Generally, the main research objective behind the development of many shell elements has been to provide an element which is versatile, robust and reliable for all possible analysis conditions. In addition, some authors (e.g. Bathe and Dvorkin [61]) have expressed the additional constraint of the element fitting into the existing framework of finite element methodology (i.e. the use of six degrees of freedom per node).

In the general shell problem, the element can be arbitrarily orientated in space and be geometrically representing a curved surface. In addition, the first derivative of deflection across the boundary of adjacent elements should be continuous. The bending components of the general shell equations require at least first order continuity in the out of plane displacements and, to maintain slope compatibility, higher second order derivatives must be introduced at the nodes. These derivatives must then be transformed to a common coordinate system so that the stiffness and mass matrices of the structure can be assembled. Having transformed the local partial derivatives, it is possible that some of them are zero as a result of the 'conservation of normals' assumption. For example, consider a part of a shell structure lying in the X-Y plane. The nodal parameter $U_{Z,Z}$ is zero and must be accounted for in the assembly of the structure's stiffness and mass matrices. In addition, a compatible element of this type has eighteen degrees of freedom per node which is far in excess of the six favoured by many researchers.

For many elements the conventional practice of ignoring the in-plane rotation when describing the shell kinematics, results in a poorly conditioned problem if the elements are nearly coplanar. A number of numerical techniques have been introduced to overcome this problem [45,46,62]. However, in the cases referenced, there are no explanations for the techniques and their performance is both problem and computer dependant.

The Finite Difference Method provides an alternative methodology which avoids the problems associated with computing accurately the geometric properties of the surface and transforming the unknown nodal derivative parameters to a common coordinate system.

In essence, expressions defining the local partial derivatives are written in terms of the adjacent nodal displacements. These are then combined with the pertinent surface geometric properties and substituted into the governing differential equations of motion. The resultant system of equations is then solved to yield the dynamic characteristics of the structure.

Although theoretically viable, the technique described above does have a number of problems associated with its implementation. Firstly, the differential equations describing the dynamic behaviour of general thin shells are extremely complex. They contain fourth order partial derivatives and require the algebraic derivation and expansion of such quantities as the second covariant derivative of the moment tensor in terms of the unknown nodal displacements. Secondly, the resultant stiffness and mass matrices will have a high bandwidth due to the problems fourth order nature and in the general case be non-symmetric. Numerical solution techniques which solve non-symmetric eigenvalue problems and incorporate a pivoting algorithm to overcome possible ill-conditioning associated with the boundary conditions would therefore need to be employed.

The application of the finite difference method to the analysis of plates and shells has been under investigation by a number of researchers [63] - [69]. In general, the work has been restricted to the static analysis of structures having a predefined geometry . This enables the governing differential equations to be simplified.

Morino, Leech and Witmer [63] - [65] have developed a tensor based methodology for the calculation of large elastic-plastic dynamically-induced deformations of general thin shells. Their program, PETROS 2, uses a combination of forward, backward and central finite difference equations which are derived from Taylor-series expansions using constant grid spacings. In addition the program is restricted to shells whose boundary conditions are uniform along each side and lie parallel to the global coordinate axes.

The Finite Difference Energy Method combines some of the advantages of both the finite element and finite difference techniques. In this approach, the surface is discretized by establishing a unique set of parametric curves and assigning node numbers to their intersecting points. In many formulations [70] - [76] these curves need not be restricted to orthogonal cases. Associated with each node is a non-overlapping subdomain which, when combined with adjacent nodes, represents the shell surface numerically. Within a subdomain, the finite difference approximations for the displacements and corresponding derivatives are substituted into the integral form of the differential equations of motion. The total potential energy of the system is computed as the sum of the potential energies summed over the set of nodal subdomains. The resulting system of equations are then solved to yield the shell's dynamic characteristics.

There are a number of advantages in formulating shell problems based on the above approach. Firstly, the integral form of the differential equation of motion contains only first and second order displacement derivatives. Therefore, a second order finite differencing scheme can be used, enabling a reduction in the bandwidths associated with the stiffness and mass matrices of the structure. Secondly, the resultant system of equations describing the dynamic behaviour of the shell will be symmetric and therefore allow the use of efficient eigenvalue/eigenvector solution techniques.

A number of shell analysis programs based on the above method have been developed. The earliest examples appearing within the literature were applied to the analysis of axisymmetric shells and are detailed in references [77] - [80]. In particular, the BOSOR series of programs developed by Bushnell [79,80] were developed with the capacity to perform linear, nonlinear, stability and vibration analyses of complex segmented, ring-stiffened shells of revolution with various wall constructions.

The application of the finite difference energy method to the analysis of shells of arbitrary geometry with the use of irregular grids was first published by Johnson [70]. In his approach, Greens Theorem is used to obtain finite difference expressions for the displacement derivatives which are required in computing the energy function. However, this application was restricted to the static analysis of shells only.

The STAGS computer program developed by Bushnell [79] provides the capability of performing a free vibration analysis on shells of arbitrary geometry. Although the program permits variable mesh spacing, the mesh lines are required to be parallel to the coordinate lines thereby restricting its use to a particular range of problems.

The development of a technique which is capable to express the partial derviatives as finite differences based on a curvilinear coordinate system and to make the finite difference method readily programmable was suggested by Lau in 1977[74].This technique was initially applied to develope the Cell collocation method for solving continuum mechanics problems by Lau and Brebbia [112].In the process of extending the applications to three dimensional potential equations[113], biharmonic equations[114] and plate bending problems[74], the name Curvilinear Finite Differences was used by one of the authors to denote such a finite difference technique.

Recently, Kwok [71,72] combined the Curvilinear Finite Difference (CFD) method introduced by Lau [74] with the general strain-displacement equations of Flugge [1]. The resulting methodology was capable of performing a linear and non-linear static analysis of general thin shells. Its application to non-shallow shells highlighted the importance of determining accurately the geometric properties of a surface and applying the complete strain-displacement equations to