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Edited by C. A. Brebbia

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Preface

The first volume of this series dealt with the Basic Principles of Boundary Elements, while the second concentrated on time dependent problems and Volume three on the Computational Aspects of the method. This volume studies the applications of the method to a wide variety of geomechanics problems, most of which are ideally suited for boundary elements demonstrating the potentiality of the technique.

Chapter 1 deals with the application of BEM to three dimensional elastodynamics soil-structure interaction problems. It presents detailed formulations for rigid, massless foundations of arbitrary shape both in the frequency and time domains. The foundations are assumed to be resting on a linearly elastic, homogeneous, isotropic half-space and be subjected to externally applied loads on obliquely incident body. The chapter reviews the major advances in soil foundation interaction presents a series of numerical results and stresses the practical application of BEM, pointing out the high accuracy and efficiency of the technique, even when using coarse mesh discretizations.

Chapter 2 presents a number of numerical results after a brief review of the basic equations governing the dynamics formulation of BEM. The general foundation stiffness problem is presented and stiffness coefficients are calculated for 3D, 2D and axisymmetric foundations. Then the response of foundations to travelling harmonic waves is studied, including the cases of embedded, strip and rectangular foundation. Another section reviews the use of time discretization BEM for the analysis of foundations. This chapter presents the state of the art on the topic of soil foundation interaction including diagrams obtained using numerical results and of utility in engineering practice.

Chapter 3 reviews the BEM formulation for consolidation problems, including Biot's method and the more practical approaches used by soil mechanicists. The formulations are described in detail and some numerical examples are presented. The merit of the chapter lies not only in the up to date presentation of all available theory but in the attempt by the author to relate together all the different approaches.

A review of the Boundary Element Model for salt water intrusion is attempted in the following chapter. This is a particularly attractive application for BEM as it involves moving boundaries. BEM models are concluded to cover a broad spectrum of salt water – intrusion problems and to be superior to finite difference or finite element models for this type of interface problem.

Chapter 5 studies a problem of interest in soil mechanics and others, i.e. the application of the BEM to the study of non-linear interface behaviour between two material regions. The non-linear interface response is modelled either by Coulomb frictional behaviour or by interface plasticity. The results presented by the authors illustrate the manner in which the non-linear phenomena at the interface contributes to the global non-linear response in the composite.

The last three chapters deal with flow through porous media. Chapter 6 presents some innovative ideas to deal with heterogeneities in these flows which is a problem for which BEM was thought to be inappropriate. The approach is based on using simple transformations and the author presents a table listing some of them. Notice that the method can be used for multizoned regions as well.

The effects of spatial distribution of soil infiltration properties and rainfall rates on the performance of a catchment area are studied in Chap. 7. Several important conclusions are obtained using recharge sources varying in time and space.

The last chapter, 8, revises the BEM formulation of unconfined groundwater flow, involving a moving, non-linear free surface as well as using the vertically integrated approach.

The chapters in this volume bring together a series of recent advances made in the applications of BEM for soil mechanics, consolidation, foundations and groundwater flow problems. Research in these problems has progressed to a degree such that the BEM can be employed as an engineering analysis tool.

Southampton, June 1987

Carlos A. Brebbia
Editor

Contents

Chapter 1

Three-Dimensional Soil-Structure Interaction by Boundary Element Methods

by D.L. Karabalis and D.E. Beskos

Abstract

The application of the Boundary Element Method to the linear three-dimensional soil-structure interaction problem is discussed. Detailed formulations for rigid, surface, massless foundations of arbitrary shape are given in both frequency and time domains. In both cases the foundations are assumed to rest on a linear elastic, homogeneous, and isotropic half-space and are subjected to either externally applied loads or obliquely incident body or surface waves. Results obtained by the above approaches as well as by other well established techniques are given in a comparison study. More general problems involving massive foundations and superstructures are also presented in the general framework of a substructure formulation.

1.1 Introduction

During the fifty years that have passed since Reissner's pioneering work [1, 2], a significant amount of research activity has been devoted to the field of dynamic soil-structure interaction (SSI). As a result of that, a number of techniques are available today for solving a variety of problems within the general area of SSI. These techniques range from the simplified to the very sophisticated and are even available commercially. However, due to the highly complex character of the real problem and our limited knowledge of the several factors which might affect its solution, all of the existing models deal with highly idealized portions of it.

In very general terms, the dynamic SSI problem consists of computing the stresses or the displacements that would occur at the contact surface between the foundation and the soil when the soil-structure system is subjected to general transient external loads or obliquely incident seismic waves. These contact stresses do not only affect the motion of the overlying structure but deform the neighboring soil as well. This soil deformability has been proven to influence substantially the overall behavior of the soil-structure system and solely accounts for the SSI phenomenon. There are cases, of course, in which the deformations of the soil are insignificant compared to the distortions within the structure itself or to the uncertainties involved in specifying these deformations, and thus one can assume that the structure is supported on a rigid soil. There exists evidence in several cases, however, such as in tall buildings supported on moderately soft soils, extensive mat foundations, or adjacent multiple footings, to name but a few, that SSI phenomena occur and are of importance. In order to account for these phenomena in some sort of a "realistic" way, a wide variety of factors should be taken into consideration. For example, with regard to the geometry, the three-dimensional character of the problem, and the arbitrary shape and arrangement of adjacent structures, are only a few of the factors which should be considered. Further, a mechanical and material characterization of the soil-structure system should take into consideration rather well-known properties of the related structural elements and soil deposits, such as concrete slabs and domes, steel frames, soil stratification, etc., as well as soil properties not yet thoroughly understood, such as nonlinear dynamic stress-strain relationships and dynamic consolidation. Discontinuities of stresses as a result of lateral separation or uplift at the contact surface between the soil and the foundation should also be considered.

This article addresses the problem of the interaction between a three-dimensional rigid, surface foundation and a homogeneous, isotropic, linear elastic half-space, where no uplift or lateral separation can occur at the contact surface between the two media. An almost complete account of the research done in this area is given in the book of Richart, Hall and Woods [3], which reviews the work up to 1968, as well as in several other updated references, e.g. [4–8, 80]. In general it has been customary to classify the SSI methods into three major groups: the "continuous" models, the "half-space approximate" models, and the "discrete finite" models.

In the first group, the "continuous" models, the dynamic stiffnesses or flexibilities of the foundation are obtained in a complex form by solving semi-analytically the governing equations of the complete mixed boundary value problem. Collins [9], Robertson [10], Gladwell [11], and others, used the above approach in order to obtain solutions for the various modes of vibration of a rigid circular disc on an elastic half-space. Veletsos and Verbic [12] also considered the problem of the rigid circular footing but on a viscoelastic half-space, thus introducing the effect of the soil material damping in addition to the radiation or geometrical damping. The solutions reported in Refs. [9–12] were obtained under the assumption of relaxed boundary conditions, i.e., the various modes of oscillation of the circular disc on the half-space are assumed to be independent from each other. The coupling effect between the tangential and rocking oscillations of a rigid circular disc was studied by Veletsos and Wei [13] and Luco and Westmann [14], while Luco and Westmann [15] considered the problem of complete bond between the soil and a strip foundation. The "continuous" models, although successful in presenting convenient semi-analytical solutions, are of small practical importance since they can only be applied to relatively simple cases involving harmonic motions, simple boundaries, and linear elastic or viscoelastic materials.

In the second general group of approaches, the "half-space approximate" models, the problem of rigid foundations of arbitrary shape is introduced. The various existing techniques that fall into this major category proceed first by discretizing the contact surface between the soil and the foundation into a number of "subregions" of simple geometrical form. This step is necessary in order to replace the unknown contact stresses under each "subregion" with an approximate distribution of stresses, usually of constant variation or with only a concentrated force at the center of the "subregion", and at the same time to make possible the application of analytical solutions, e.g., from Lamb's problem [16], to express the stresses and displacements under each "subregion". Finally, appropriate compatibility and equilibrium conditions are applied among all the "subregions" in order to simulate the rigid body motion of the foundation. This basic methodology was introduced as early as 1965 by Lysmer [17], who studied the vertical flexibility of a circular rigid disc on an elastic half-space by considering several uniformly loaded concentric rings. Numerous other applications of the method have been reported for various problems in the general area of SSI. For example, the works of Elorduy, Nieto and Szekely [18], Wong and Luco [19], Gaul [20], Kitamura and Sakurai [21, 22], Adeli et. al. [23], and Hamidzadeh-Eraghi and Grootenhuis [24], are among the many analyzing the problem of a single surface foundation of arbitrary shape under externally applied forces. Cross-interaction effects between adjacent foundations were also studied by Savidis and Richter [25] and Gantayat and Kamil [26], among others. Wong and Luco [27, 28] considered the effect of oblique seismic waves on surface, rigid, rectangular foundations, and Bielak and Coronato [29] the motion of two rectangular foundations on a viscoelastic half-space under the influence of oblique SH and Rayleigh waves. The complete problem of an entire structure, supported on a set of rigid surface foundations, and subjected to nonvertically incident seismic waves was also formulated and solved by Werner, Lee, Wong and Trifunac [30] and Luco and Wong [31].

The third major group of methods applied to SSI, the "discrete finite" models, consists of the Finite Element Method (FEM) and the Finite Difference Method (FDM), the two most widely used numerical methods for general wave propagation problems, with the former being the most popular of the two. From the many reported applications of the FEM to general SSI problems, very few have dealt with the three-dimensional rigid surface foundation case, e.g., Roesset and Gonzalez [32], Gupta, Penzien, Lin and Yeh [33], and Dasgupta and Rao [34]. An SSI analysis using the FEM appears, in principle, to be very effective because of the distinct advantages that this method presents in handling complex foundation geometries and soil inhomogeneities. In addition, a time domain FEM analysis, through a step-by-step integration, provides the means for studying nonlinear SSI problems. However, the use of the FEM in SSI presents serious drawbacks due to the fact that the semi-infinite soil medium is represented by a finite size model and thus wave reflecting boundaries are artificially introduced. This inherent deficiency can be compensated for in part by the use of very large models, or non-reflecting (transmitting) boundaries [35, 36], or special infinite elements [37, 38]. None of these improvements succeeds in completely eliminating the problem, and in fact they complicate the application of the FEM and make it highly uneconomical. Hybrid finite element models for three-dimensional SSI have also been reported, e.g., [33, 39–45]. Detailed comparisons of the two basic methods for solving SSI problems, namely, the "half-space approximate" method and the FEM, have been reported by Hadjian, Luco and Tsai [46], favoring the former method, and by Seed, Lysmer and Hwang [47] favoring the latter.

Several applications of the FDM in the general area of SSI have also been reported [48–52]. This method, like the FEM, presents us with the built-in problem of the artificially introduced wave reflecting boundaries, and attempts have been made to correct it by the use of specially developed two-dimensional nonreflecting boundaries [51–52]. However, due to its inferior handling of the complicated geometries frequently encountered in SSI problems, the popularity of the FDM in this area remains limited.

In this brief literature review and classification of methods currently applied to the SSI analysis, the discussion of the Boundary Element Method (BEM) was left for last since it constitutes the main subject of this article. The BEM is the latest addition to the group of major numerical methods used for dynamic SSI analyses, and appears to be ideally suited for modelling the half-space soil medium, particularly in three-dimensional problems. The reason, of course, is that this method requires discretization of only the boundary of the domain of interest, which in the present case is usually the contact surface between the soil and the foundation, and takes into account automatically the radiation condition due to the use of the fundamental singular solutions (Green's functions). Further, the immediate consequences of requiring a discretization only at the surface are, first, that the dimensions of the problem are reduced by one and, second, a minimum amount of discretization is required. These are distinct advantages over "domain" type methods, such as the FEM and the FDM, which require a discretization of both the interior of the domain of interest and its surface, and artificial nonreflecting boundaries.

To the writers' best knowledge, Dominguez [53, 54] was the first to apply the BEM in order to compute, in a frequency domain formulation, the impedances of two- and three-dimensional surface and embedded, rigid, rectangular foundations resting on a uniform linear elastic half-space. Ottenstreuer and Schmid [55] and Ottenstreuer [56], following the same approach, studied, respectively, the problems of rigid, surface foundations and of cross-interaction between two rigid, surface, rectangular foundations. Apsel [57] obtained the dynamic stiffnesses of rigid cylindrical foundations embedded in a uniform or layered viscoelastic half-space using an indirect BEM. Both relaxed and non-relaxed boundary conditions were investigated in Refs. [53–57, 79]. Recently Wolf and Darbre [58, 59] studied the application of three BEM formulations, i.e., the weighted-residual technique and the direct and indirect BEM, to the problem of a two-dimensional foundation embedded in a layered elastic half-plane. A time domain BEM for three-dimensional SSI analysis was first reported by Karabalis and Beskos [60]. This method, which uses step-by-step integration in time, has also been applied to the interaction problems of a linear elastic half-space with rigid surface [60], rigid embedded [5, 61], and flexible surface foundations [62].

Hybrid methods involving the FEM and the BEM have also been reported, e.g., [63, 64]. In the SSI case, the BEM is used to describe the unbounded exterior domain and the FEM the bounded interior one, thus taking full advantage of the merits of both methods. This idea, which is similar to the hybrid methods of Refs. [33, 39–45], has been utilized by various workers in the field such as Varadarajan and Singh [65], Mita and Takanashi [66], and Gaitanaros and

Karabalis [78], to name but a few in the frequency domain, and Karabalis and Beskos [62] in the time domain.

It should be emphasized at this point that, in contrast to all the other frequency domain methods, the general time domain methodology reported in Refs. [5, 60–62] presents a natural and direct way of dealing with transient problems and, in addition, can form the basis for an extension to nonlinear SSI analysis where nonlinearities could occur in the soil and/or the structure. Of course a nonlinear SSI analysis can be done, in principle, by a direct application of the time domain FEM. However, due to its deficiencies in simulating unbounded regions, the use of this method in a true three-dimensional SSI analysis is associated with prohibitive high costs and frequently with inaccuracies. The time domain approach of Veletsos and Verbic [67], which is restricted to circular foundations only, is based on impulse response functions in a convolution formulation which precludes its extension to nonlinear soil problems.

In the following sections the problem of the dynamic response of a three-dimensional rigid, surface, massless foundation on a linear elastic, homogeneous and isotropic half-space due to both externally applied loads and obliquely incident seismic waves, is formulated using the BEM. Both the frequency and the time domain approaches are presented in detail following the developments reported in Refs. [53, 54] and [5, 60], respectively. An extension of the massless foundation problem to more general problems involving massive foundations and/or superstructures is presented on the basis of a soil-structure interaction superposition principle. Finally, comparisons between the time domain and frequency domain BEM as well as other well known methodologies are given and conclusions on their accuracy and efficiency are drawn.

1.2 Field Equations

In this section the classical integral formulation of the elastodynamic problem, in both the time and the Fourier transformed or frequency domain, is reviewed. The following developments are based on the book by Eringen and Suhubi [68], for the time domain, and the work of Cruse and Rizzo [69], for the frequency domain. The standard index notation has been adopted where summation is assumed over repeated indices, commas indicate spatial differentiation, dots indicate differentiation with respect to time t, Greek subscripts take the values 1 and 2, and Latin subscripts take the values 1 to 3.

The governing equations, in terms of displacements, for a homogeneous, isotropic, linear elastic solid occupying a regular region R with a surface B can be written in the form

$$(c_1^2 - c_2^2)u_{i,ij} + c_2^2 u_{j,ii} + f_j = \ddot{u}_j, \tag{1}$$

where f_j is the body force vector per unit mass, the dilatational and shear wave velocities are given, respectively, as

$$c_1^2 = (\lambda + 2\mu)/\rho, \quad c_2^2 = \mu/\rho, \tag{2}$$

with μ and λ being the Lame constants, ρ being the mass density, and

$$u_i = u_i(\boldsymbol{x}, t), \tag{3}$$

where $\boldsymbol{x}$ denotes the position vector. The stresses and the displacements should satisfy the following boundary conditions:

$$\begin{aligned} t_{(\boldsymbol{n})i}(\boldsymbol{x}, t) &= p_i(\boldsymbol{x}, t), \quad \boldsymbol{x} \in B_t \\ u_i(\boldsymbol{x}, t) &= q_i(\boldsymbol{x}, t), \quad \boldsymbol{x} \in B_u, \end{aligned} \tag{4}$$

where t_{ij} is the stress tensor, $\boldsymbol{n}$ is the normal vector on a differential element of the surface B, and $B_t + B_u = B$; and initial conditions:

$$\begin{aligned} u_i(\boldsymbol{x}, 0^+) &= u_{0i}(\boldsymbol{x}), \quad \boldsymbol{x} \in R \cup B \\ \dot{u}_i(\boldsymbol{x}, 0^+) &= v_{0i}(\boldsymbol{x}), \quad \boldsymbol{x} \in R \cup B. \end{aligned} \tag{5}$$

The constitutive equation for the above solid reads

$$t_{ij} = \rho(c_1^2 - 2c_2^2)u_{m,m}\delta_{ij} + \rho c_2^2(u_{i,j} + u_{j,i}), \tag{6}$$

where δ_{ij} is the Kronecker delta, and the stress vector is given by

$$t_{(\boldsymbol{n})i} = t_{ij}n_j. \tag{7}$$

1.2.1 Time Domain Integral Representation

In order to establish an integral representation solution of Eq. (1), it is first necessary to specify the required fundamental solution that will be used. In this formulation, the fundamental singular solution of Eq. (1) in an infinite solid medium due to a concentrated body force will serve in this capacity. Such a body force can be expressed as

$$\rho \boldsymbol{f}(\boldsymbol{x}, t) = f(t)\delta(\boldsymbol{x} - \boldsymbol{\xi})\boldsymbol{e}, \tag{8}$$

where $\boldsymbol{x}$ and $\boldsymbol{\xi}$ are points in a body of infinite extent, δ is the Dirac delta function, $\boldsymbol{e}$ is the direction in which the above force is applied, and $f(t)$ is its time variation. Application of Eq. (8) as a body force in Eq. (1) will yield the response of the infinite medium in the form

$$u_i = u_{ij}^0 e_j, \tag{9}$$

where the second order displacement tensor, usually called the fundamental singular solution of the elastodynamic equations or Stokes' displacement tensor, is given by

$$\begin{aligned} u_{ij}^0(\boldsymbol{x}, t; \boldsymbol{\xi} | f) = \frac{1}{4\pi\rho}\Bigg\{&\left(\frac{3r_ir_j}{r^3} - \frac{\delta_{ij}}{r}\right)\int_{c_1^{-1}}^{c_2^{-1}} \lambda f(t - \lambda r)\, d\lambda \\ &+ \frac{r_ir_j}{r^3}\left[\frac{1}{c_1^2}f\left(t - \frac{r}{c_1}\right) - \frac{1}{c_2^2}f\left(t - \frac{r}{c_2}\right)\right] + \frac{\delta_{ij}}{rc_2^2}f\left(t - \frac{r}{c_2}\right)\Bigg\}, \end{aligned} \tag{10}$$

where

$$r_i = x_i - \xi_i, \quad r^2 = (x_i - \xi_i)(x_i - \xi_i). \tag{11}$$

The Stokes' displacement tensor expresses the i-component of the displacement that occurs at point $\boldsymbol{x}$ and time t due to a concentrated force of magnitude $f(t)$ applied at point ξ in the j-direction. The function $f(t-s)$, as used in Eq. (10), is assumed to be time retarded, i.e., it is non-zero only if $t-s>0$. The Stokes' stress tensor related to u_{ij} can be obtained through substitution of Eq. (10) into the constitutive Eq. (6). Following the above notation the Stokes' stress tensor can also be written in the form

$$t^0_{ijk} = t^0_{ijk}(\boldsymbol{x}, t; \xi | f). \tag{12}$$

The pair of fundamental singular solutions $[u^0_{ij}, t^0_{ijk}]$ possesses the properties of causality, and translation, and is called the Stokes' state of quiescent past [68].

Using Stokes' state of quiescent past as one of the two distinct elastodynamic states in Betti's reciprocal theorem, the other one being the actual state, one can derive a solution of the problem posed by Eqs. (1), (4), and (5) in the form of Love's integral representation

$$\begin{aligned}\varepsilon(\xi)u_k(\xi, t) = & \int_B \{u^0_{ik}[\boldsymbol{x}, t, \xi | t_{(\boldsymbol{n})i}(\boldsymbol{x}, t)] - t^0_{(\boldsymbol{n})ik}[\boldsymbol{x}, t; \xi | u_i(\boldsymbol{x}, t)]\}\, dB(\boldsymbol{x}) \\ & + \rho \int_R u^0_{ik}[\boldsymbol{x}, t; \xi | f_i(\boldsymbol{x}, t)]\, dR(\boldsymbol{x}) \\ & + \rho \int_R [v_{0i}(\boldsymbol{x}) U_{ik}(\boldsymbol{x}, t; \xi) + u_{0i}(\boldsymbol{x}) \dot{U}_{ik}(\boldsymbol{x}, t; \xi)]\, dR(\boldsymbol{x}),\end{aligned} \tag{13}$$

where

$$\varepsilon(\xi) = \begin{cases} 1 & \text{if } \xi \in R \\ \frac{1}{2} & \text{if } \xi \in B \\ 0 & \text{if } \xi \in R \cup B, \end{cases} \tag{14}$$

$$t^0_{(\boldsymbol{n})ik} = t^0_{ijk} n_j, \tag{15}$$

is the traction tensor of the Stokes' state. The tensor U_{ik}, also called Green's tensor, can be obtained as a special case of Stokes' displacement tensor by setting $f(t) = \delta(t)$ in Eq. (10).

For problems in SSI analysis it is usually assumed that both the body forces and the initial conditions are zero, and thus Eq. (13), written for points ξ on the boundary, is simplified to

$$\tfrac{1}{2}u_k(\xi, t) = \int_B \{u^0_{ik}[\boldsymbol{x}, t; \xi | t_{(\boldsymbol{n})i}(\boldsymbol{x}, t)] - t^0_{(\boldsymbol{n})ik}[\boldsymbol{x}, t; \xi | u_i(\boldsymbol{x}, t)]\}\, dB(\boldsymbol{x}). \tag{16}$$

Equation (16) relates boundary displacements and tractions and thus it constitutes the appropriate boundary integral equation (BIE) to be used in a time domain formulation of the foundation problems under consideration.

1.2.2 Frequency Domain Integral Representation

The Fourier transform of a function $f(\boldsymbol{x}, t)$ is defined as

$$\bar{f}(\boldsymbol{x}, w) = F\{f(\boldsymbol{x}, t)\} = \int_{-\infty}^{+\infty} f(\boldsymbol{x}, t) e^{-iwt}\, dt, \tag{17}$$

where w is the Fourier transform parameter, i.e., the frequency. Using this definition the general elastodynamic problem stated by Eqs. (1), (4), and (5) can be transformed into a static-like problem with respect to the frequency parameter w. Thus, assuming quiescent initial conditions as in the previous section, i.e., $u_{0i}(\boldsymbol{x}) = v_{0i}(\boldsymbol{x}) = 0$, the governing equations are transformed into

$$(c_1^2 - c_2^2)\bar{u}_{i,ij} + c_2^2\bar{u}_{j,ii} + \bar{f}_j = -w^2\bar{u}_j, \tag{18}$$

and similarly the boundary conditions, Eq. (4), and the constitutive relations, Eq. (6), become, respectively,

$$\begin{aligned} \bar{t}_{(\boldsymbol{n})i}(\boldsymbol{x}, w) &= \bar{t}_{ij}n_j = \bar{p}_i(\boldsymbol{x}, w), \quad && \boldsymbol{x} \in B_t \\ \bar{u}_i(\boldsymbol{x}, w) &= \bar{q}_i(\boldsymbol{x}, w), && \boldsymbol{x} \in B_u, \end{aligned} \tag{19}$$

$$\bar{t}_{ij} = \rho(c_1^2 - 2c_2^2)\bar{u}_{m,m}\delta_{ij} + \rho c_2^2(\bar{u}_{i,j} + \bar{u}_{j,i}). \tag{20}$$

Following the same basic steps as in the time domain case, one should specify the fundamental solutions that will be used before proceeding with the integral representation solution of Eq. (18). The fundamental solution of Eq. (18) for an infinite region subjected to a concentrated body force is again the natural choice. In this case the concentrated body force becomes the Fourier transform of Eq. (8), i.e.,

$$\rho\bar{f}_i(\boldsymbol{x}, w) = \bar{f}(w)\delta(\boldsymbol{x} - \xi)e_i, \tag{21}$$

and the fundamental solution of Eq. (18) can be expressed in the form

$$\bar{u}_i = \bar{u}_{ij}^* e_j. \tag{22}$$

The second order displacement tensor appearing in Eq. (22) is given by

$$\bar{u}_{ij}^*(\boldsymbol{x}, \xi, w) = \frac{\bar{f}(w)}{4\pi\rho c_2^2}(\psi\delta_{ij} - \chi r_{,i}r_{,j}), \tag{23}$$

where

$$\chi = \left(-\frac{3c_2^2}{w^2r^2} + \frac{3c_2}{\mathrm{i}wr} + 1\right)\frac{\mathrm{e}^{-\mathrm{i}wr/c_2}}{r} - \left(\frac{c_2^2}{c_1^2}\right)\left(-\frac{3c_1^2}{w^2r^2} + \frac{3c_1}{\mathrm{i}wr} + 1\right)\frac{\mathrm{e}^{-\mathrm{i}wr/c_1}}{r} \tag{24}$$

$$\psi = \left(-\frac{c_2^2}{w^2r^2} + \frac{c_2}{\mathrm{i}wr} + 1\right)\frac{\mathrm{e}^{-\mathrm{i}wr/c_2}}{r} - \left(\frac{c_2^2}{c_1^2}\right)\left(-\frac{c_1^2}{w^2r^2} + \frac{c_1}{\mathrm{i}wr}\right)\frac{\mathrm{e}^{-\mathrm{i}wr/c_1}}{r}. \tag{25}$$

Insertion of Eq. (23) into Eq. (20) yields the traction companion tensor

$$\begin{aligned} \bar{t}_{ij}^*(\boldsymbol{x}, \xi, w) = \frac{\bar{f}(w)}{4\pi}\Bigg[&\left(\frac{d\psi}{dr} - \frac{\chi}{r}\right)\left(\delta_{ij}\frac{\partial r}{\partial n} + r_{,j}n_i\right) - \frac{2}{r}\chi\left(n_jr_{,i} - 2r_{,i}r_{,j}\frac{\partial r}{\partial n}\right) \\ &- 2\frac{d\chi}{dr}r_{,i}r_{,j}\frac{\partial r}{\partial n} + \left(\frac{c_1^2}{c_2^2} - 2\right)\left(\frac{d\psi}{dr} - \frac{d\chi}{dr} - \frac{2\chi}{r}\right)r_{,i}n_j\Bigg]. \end{aligned} \tag{26}$$

The fundamental solution pair $[\bar{u}_{ij}^*, \bar{t}_{ij}^*]$, as defined above, possesses the property of space translation.

By utilizing the transform domain form of Betti's reciprocal theorem, where one of the two required states is the $[\bar{u}_{ij}^*, \bar{t}_{ij}^*]$, the other one being the actual state, one can obtain a solution to the Fourier transformed elastodynamic problem in the form of the integral identity

$$\varepsilon(\xi)\bar{u}_j(\xi) = \int_B \bar{t}_{(n)i}(\boldsymbol{x})\bar{u}'_{ji}(\boldsymbol{x},\xi,w)\,dB(\boldsymbol{x}) - \int_B \bar{u}_i(\boldsymbol{x})\bar{t}'_{ji}(\boldsymbol{x},\xi,w)\,dB(\boldsymbol{x})$$
$$- \int_R \rho\bar{f}_i(\boldsymbol{x},w)\bar{u}'_{ji}(\boldsymbol{x},\xi,w)\,dR(\boldsymbol{x}), \tag{27}$$

where

$$\bar{u}'_{ij} = \bar{u}^*_{ij}/\bar{f}(w) \quad \text{and} \quad \bar{t}'_{ij} = \bar{t}^*_{ij}/\bar{f}(w). \tag{28}$$

Finally, on the assumption of zero body forces and points ξ on the boundary B, Eq. (27) is reduced to

$$\tfrac{1}{2}\bar{u}_j(\xi) = \int_B \bar{t}_{(n)i}(\boldsymbol{x})\bar{u}'_{ji}(\boldsymbol{x},\xi,w)\,dB(\boldsymbol{x}) - \int_B \bar{u}_i(\boldsymbol{x})\bar{t}'_{ji}(\boldsymbol{x},\xi,w)\,dB(\boldsymbol{x}) \tag{29}$$

which is the restraint equation between boundary tractions and displacements, i.e., the BIE in the Fourier transformed domain. Equation (29) provides the solution to a given dynamic problem in terms of the frequency parameter w. The time domain response can then be obtained through a Fourier synthesis of a sequence of frequency dependent solutions.

An analytic solution of the previously established boundary integral equations in the time, Eq. (16), or the frequency, Eq. (29), domains is almost impossible even for simple geometries and time variations of the related functions. The numerical treatment of Eqs. (16) and (29) is the subject of the next section.

1.3 Numerical Implementation

Toward a numerical solution of the boundary integral Eqs. (16) and (29), a spatial discretization is necessary in both the time and the frequency domain approaches. However, in contrast to the static-like frequency domain problem, the time domain formulation requires an additional discretization in time. The numerical procedures described in this section are following the developments reported by Karabalis [5], and Karabalis and Beskos [60] for the time domain, and Cruse [70] for the frequency domain case.

1.3.1 Time Domain Approach

The time domain BEM of Refs. [5, 60] consists of two basic steps: (i) a discretization of the real time axis into a sequence of equally spaced time intervals is applied, and further, the variation of the displacements and tractions over each time interval is assumed to be constant, and (ii) the boundary B of the domain of interest is discretized into a number of rectangular elements over each of which a constant distribution of displacements and tractions is considered. On the basis of these discretizations a time stepping solution of Eq. (16) can be established for the boundary displacements and tractions over each rectangular element and for each time step.

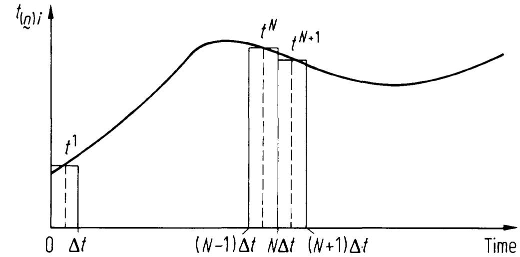

Fig. 1. Approximation of the surface traction $t_{(n)i}$ by a sequence of rectangular pulses (after Karabalis [5])

To illustrate the above outlined procedure, one might consider the time discretization of the continuous traction vector into a sequence of rectangular impulses as shown in Fig. 1. This approximation can also be expressed, for the time interval $(q-1)\Delta t < t < q\Delta t$, as

$$t_{(n)i}(x,t) \approx t^q_{(n)i}(x)\{H[t-(q-1)\Delta t] - H(t-q\Delta t)\}, \tag{30}$$

where

$$t^q_{(n)i} = t_{(n)i}[x,(q-0.5)\Delta t], \tag{31}$$

and represents the intensity of the rectangular impulse at time $t = (q-0.5)\Delta t$. Substituting the forcing function $f(t)$ in Eq. (10) by $t_{(n)i}(x,t)$ as it is expressed in Eq. (30), e.g., for $q = 1$, will yield the following time discretized expression for the Stokes' displacement tensor:

$$\begin{aligned} u^0_{ik}(x,t;\xi\,|\,t^1_{(n)i}) = \frac{1}{4\pi\rho}\Bigg\{&\left(\frac{3r_ir_k}{r^5}-\frac{\delta_{ik}}{r^3}\right)\left[H\left(t-\frac{r}{c_1}\right)F_1 - H\left(t-\frac{r}{c_2}\right)F_2\right] \\ &\cdot t^1_{(n)i}(\xi) + \frac{1}{c_1^2}\frac{r_ir_k}{r^3}H\left(t-\frac{r}{c_1}\right)t_{(n)i}\left(\xi,t-\frac{r}{c_1}\right) \\ &+\frac{1}{c_2^2}\left(\frac{\delta_{ik}}{r}-\frac{r_ir_k}{r^3}\right)H\left(t-\frac{r}{c_2}\right)t_{(n)i}\left(\xi,t-\frac{r}{c_2}\right)\Bigg\}, \end{aligned} \tag{32}$$

where

$$F_\beta = \begin{cases} [t^2-(r/c_\beta)^2]/2, & \text{if } 0 < t-(r/c_\beta) \leqslant \Delta t \\ [2t(\Delta t)-(\Delta t)^2]/2, & \text{if } \Delta t < t-(r/c_\beta),\ \beta = 1, 2. \end{cases} \tag{33}$$

A similar discretization in time of the boundary displacement vector $u_i(x,t)$, and direct substitution of it into the expression of the Stokes' stress tensor, indicated by Eq. (12), will result in a time discretized form of the Stokes' stress tensor [5].

The spatial discretization scheme utilized in 3-D SSI, for example, is shown in Fig. 2a, in which the contact surface between an arbitrarily shaped foundation and the half-space is discretized into an M number of rectangular elements.